Quasilocal quantities for GR and other gravity theories

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Abstract.   From a covariant Hamiltonian formulation, by using symplectic ideas, we obtain certain covariant boundary expressions for the quasilocal quantities of general relativity and other geometric gravity theories. The contribution from each of the independent dynamic geometric variables (the frame, metric or connection) has two possible covariant forms associated with the selected type of boundary condition. The quasilocal expressions also depend on a reference value for each dynamic variable and a displacement vector field. Integrating over a closed two surface with suitable choices for the vector field gives the quasilocal energy, momentum and angular momentum. For the special cases of Einstein's theory and the Poincaré Gauge theory our expressions are similar to some previously known expressions and give good values for the total ADM and Bondi quantities. We apply our formalism to black hole thermodynamics obtaining the first law and an associated entropy expression for these general gravity theories. For Einstein's theory our quasilocal expressions are evaluated on static spherically symmetric solutions and compared with the findings of some other researchers. The choices needed for the formalism to associate a quasilocal expression with the boundary of a region are discussed.

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1. Introduction

The fundamental quantities energy-momentum, and angular momentum of the gravitational field are elusive. Globally, for spacetimes which are asymptotically flat (or even anti-DeSitter), there are well defined values for the *total* energy, etc., given by the Bondi and ADM expressions integrated over spheres at null infinity and spatial infinity, values which are directly related to quantities that can be physically measured by distant observers. However, unlike all other fields, for the gravitational field these quantities have no well defined *local density* since, according to the *equivalence principle*, an observer cannot detect any features of the gravitational field at a point. It has even been argued that the proper energy-momentum of the gravitational field is only total; that it cannot (or should not) be localized (see, e.g., [1] p 467).

Localization is certainly possible if it is simply understood to mean "find some way of dividing up the total". In particular each of the many proposed *pseudotensors* provides for such a localization. But pseudotensors depend on the reference system, so they actually provide for many different localizations each of which includes some rather arbitrary unphysical content; thus one can arrange, at any selected point, for almost any value: positive, zero or even negative. Certain positive energy proofs (e.g., [2, 3, 4]) may be a better alternative: each gives localizations which are *positive*. But actually these "positive localizations" are not truly "local", for they divide up the total in a way which depends on the configuration nonlocally, since they each depend on the solution of an elliptic equation which in turn depends on the field values everywhere.

Appreciating the fundamental nonlocality of the gravitational interaction yet believing in the basically local nature of physical interactions led to the idea of *quasilocal quantities*: quantities that take on values associated with a compact orientable spatial 2-surface. Expressions for quasilocal quantities in the context of Einstein's general relativity theory have been proposed from many perspectives including null rays [5], twistors [6], a fixed background [7], symplectic reduction [8], spinors [9, 10, 11, 12], a 2+2 "Hamiltonian" [13], Hamilton-Jacobi [14] and Ashtekar variables [15]. Not surprisingly the various definitions generally give different results [16, 10].

Lists of criteria to be satisfied by a quasilocal energy have been devised, see, e.g., [17, 10]. Usually it is required that it should vanish for flat spacetime, give reasonable values for weak fields and spherically symmetric solutions, and should approach the ADM and Bondi values in the appropriate limits. Opinions differ concerning whether quasilocal energy must be non-negative, but in any case such criteria are not sufficiently restrictive—in fact there remain an infinite number of possibilities [10].

Here we offer a comprehensive presentation of some new ideas regarding quasilocal energy, many of which were first developed in [18] and briefly reported in [19]. In our approach we begin with the idea that energy is naturally associated with time translation

and is thus given by the value of the time translation generator: the Hamiltonian. Hence good expressions for quasilocal quantities (energy, etc.) should be based not only on the variational principle and Noether's theorem but especially on the canonical Hamiltonian procedure, a viewpoint shared by some other investigators, e.g., [8, 14].

The fundamental feature that distinguishes our Hamiltonian based expressions for quasilocal quantities (aside from the fact that we treat rather general geometric gravity theories) is that they are 4-dimensionally covariant. We believe this is an essential quality for a physically meaningful (observer independent) definition of quasilocal quantities appropriate to a covariant gravity theory. Technically, our *covariant* Hamiltonian formulation [20] is obtained by using differential form techniques [21], in lieu of the usual spacetime splitting with its associated loss of manifest 4-covariance.

For dynamic geometry theories the Hamiltonian has a special form connected with its role as the generator of displacements along a timelike vector field N. Noether's theorem applied to local translations along a vector field reveals that the Hamiltonian 3-form (density) has the special form $\mathcal{H}(N) \equiv$ (terms proportional to field equations) $+d\mathcal{B}(N)$. Consequently, the Hamiltonian $H(N) = \int_{\Sigma} \mathcal{H}(N)$, which displaces a finite spacelike region Σ along N, has a value (on a solution to the field equations) given just by $\oint_{\partial\Sigma} \mathcal{B}(N)$, the integral of the 2-form $\mathcal{B}(N)$ over the boundary of the spatial region. Thus this boundary integral will determine the value of the quasilocal quantities. Although the Hamiltonian field equations themselves do not depend on the boundary term, the expression for $\mathcal{B}(N)$ (unlike the case for other Noether conserved currents) is nevertheless restricted by the Hamiltonian variational principle.

To fix the Hamiltonian boundary term we consider the variation of the Hamiltonian density. In general we get an expression of the form $\delta\mathcal{H}(N) =$ (field equation terms) $+d\mathcal{C}(N)$. The total differential gives rise to a boundary integral term in the variation of the Hamiltonian. As Regge and Teitelboim have nicely explained [22], it is necessary that the boundary term in the variation of the Hamiltonian vanish (only then are the functional derivatives well defined). This is not a problem for finite regions — if we fix the appropriate quantities on the boundary. However, when we consider the limit $r \to \infty$, the boundary term $\oint \mathcal{C}(N)$ for gravity theories does not vanish asymptotically in general. (In particular this is in fact the case for Einstein's theory.) To compensate for this the $\mathcal{B}(N)$ term in the Hamiltonian needs to be adjusted. In this way the form of $\mathcal{B}(N)$ at infinity is constrained and the value of the total conserved quantities fixed.

This argument was applied to the Poincaré Gauge theory (PGT) and even more general gravitational theories for both asymptotically flat spaces and asymptotically constant curvature spaces by one of us [20, 23] with important improvements by Hecht [24]. Expressions for the Hamiltonian boundary integrand which give the correct total conserved energy, momentum and angular momentum at spatial infinity [25] and at future null infinity [26] were obtained. These same boundary expressions can also

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be applied to a finite region to give quasilocal values. But the very form of Hecht's improvement helped to show the way to other equally valid expressions.

A comprehensive and systematic investigation of the role of total differential terms (i.e., boundary terms) in both the Lagrangian and Hamiltonian variational principle is really needed. Years ago Kijowski emphasized the importance of symplectic methods [27] in such investigations. More recently there has been increasing recognition of the importance of symplectic methods and of boundary terms in variational principles; we have benefited from the progress made by several workers [28, 29, 30, 31, 32, 33, 14].

A nice mathematical theory has been developed, which purports to incorporate the fundamental principle of physical interactions, and is especially well adapted to our problem. This is the theory of *symplectic relations* first proposed by Tulczyjew [34] and developed with Kijowski [35]. (Later Kijowski and coworkers applied it to gravity theories, e.g. [8, 36]. More recently Wald [37] has independently obtained some nice symplectic results.) In this treatment the (normally neglected) boundary term in the variational principle reflects the *symplectic structure* which, in turn reveals what physical variables are the "control" variables (held fixed) and which are the "response" variables (determined by the physical system).

We shall use this theory to better understand both the general dynamic geometry Lagrangian variational principle and its associated boundary terms and the covariant Hamiltonian formulation. Hence the symplectic formulation is a key principle underlying our work. With its aid we will recognize the symplectic structure of the boundary term $\mathcal{C}(N)$ in the variation of the Hamiltonian and be guided to select those Hamiltonian boundary terms $\mathcal{B}(N)$ which give rise to a "covariant" symplectic structure for $\mathcal{C}(N)$.

The application of these ideas, the covariant Hamiltonian procedure along with a covariant symplectic boundary variation, determines the expressions for the quasilocal quantities of a gravitational system. In this way we obtain a Hamiltonian 3-form $\mathcal{H}(N)$, which generates the evolution along the vector field N, and which includes, for each independent variable (e.g., the connection, the coframe, the metric), one of two possible covariant boundary terms depending whether the field or its conjugate momenta is held fixed (controlled) on the boundary. Then the variation of the Hamiltonian, in addition to the field equations, includes a boundary term with a covariant *symplectic* structure which reflects the choice of control mode. For each geometric field (metric, frame, connection) we can control either the field value or the momentum (essentially, Dirichlet or Neumann boundary conditions). Each choice gives rise to a different quasilocal value. Thus, as in thermodynamics, there are several different kinds of "energy", each corresponds to the work done in a different (ideal) physical process.

The Hamiltonian boundary terms are the expressions for the quasilocal quantities. They determine the value of the Hamiltonian. The physical meaning depends on the displacement: energy for a time translation, momentum for a space translation, and

angular momentum for a rotation. (Although our Hamiltonian formalism apparently presumes a timelike displacement, this is really no limitation, for the Hamiltonian is linear in the displacement vector field. Hence considering the difference between two suitably chosen timelike displacements gives the meaning of the Hamiltonian for spacelike displacements such as translations and rotations.)

A noteworthy feature of our expressions is that they have an explicit dependence on a selected *reference configuration.* The reference configuration has a simple meaning: when the dynamic fields have the reference configuration values on the boundary then all quasilocal quantities vanish. In particular, the reference configuration determines the "zero" of energy.

We present the formalism for rather general geometric gravity theories. Specifically gravity theories which can have an independent metric and connection. The connection need not be metric compatible nor symmetric. The field equations are presumed to follow from a Lagrangian which may depend in any way on the metric, the curvature, the torsion and non-metricity tensors (but *not* on their derivatives). This general geometric approach has proved to be a good guide for specific and more specialized theories. Our expressions are easily restricted to special cases such as Riemann-Cartan, Riemannian or teleparallel geometries. For Einstein's theory and the Poincaré gauge theory (for asymptotically flat *or* asymptotically constant curvature solutions) our expressions are related to previously known expressions (e.g., [22, 38, 39, 40, 41]) and give good values for the total conserved quantities at spatial [25] *and* future null infinity [26].

The outline of this work is: in section 2, we give the covariant Hamiltonian analysis of a general geometric gravitational theory in the language of differential forms. In section 3 our requirements and expressions, based on the covariant-symplectic Hamiltonian, for quasilocal quantities are presented. We apply the ideas to black hole thermodynamics obtaining the generalized first law and an expression for entropy for these general theories in section 4. The specialization of our expressions to certain gravity theories is discussed in section 5. We restrict our relations to general relativity, apply them to spherically symmetric solutions and compare them with some other authors' results in section 6. Finally our concluding discussion considers the various choices in selecting a Hamiltonian boundary term-quasilocal expression, noting in particular how this scheme provides for an orderly system relating the choices to physically meaningful properties on the boundary of a spatial region.

2. Covariant Hamiltonian formalism

In this section we briefly recount the relevant features of the covariant Hamiltonian formalism [20, 23]. We consider quite general dynamic geometry theories of the metric-affine type (see, e.g., [42]) as well as some important special cases. The possible

geometric potentials are the *metric coefficients* $g_{\mu\nu}$ the *coframe* one-form ϑ^α and the *connection* one-form $\omega^\alpha{}_\beta$. The corresponding field strengths are the *non-metricity* one-form

$$Dg_{\mu\nu} := dg_{\mu\nu} - \omega^\gamma{}_\mu g_{\gamma\nu} - \omega^\gamma{}_\nu g_{\mu\gamma}, \tag{1}$$

the *torsion* 2-form

$$\Theta^\alpha := D\vartheta^\alpha := d\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta, \tag{2}$$

and the *curvature* 2-form

$$\Omega^\alpha{}_\beta := d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta. \tag{3}$$

The usual approach is that (second order) dynamical equations are presumed to follow from a variational principle. We presume that the Lagrangian density is a scalar valued 4-form depending only on the gauge potentials and their first differentials: $\mathcal{L}^{(2)} = \mathcal{L}^{(2)}(g, \vartheta, \omega; dg, d\vartheta, d\omega)$. (Note that we have, in a very natural way, excluded higher derivatives of the fields. However, there is really no obstacle in principle to extending our formalism to include them, say via the expense of introducing extra fields and Lagrange multipliers.) In view of the invariance under local Lorentz transformations the more covariant form: $\mathcal{L}^{(2)}(g, \vartheta; Dg, \Theta, \Omega)$ has advantages. Instead of this usual 2nd order type Lagrangian 4-form we use a "first order" Lagrangian 4-form:

$$\mathcal{L} := Dg_{\mu\nu} \wedge \pi^{\mu\nu} + \Theta^\alpha \wedge \tau_\alpha + \Omega^\alpha{}_\beta \wedge \rho_\alpha{}^\beta - \Lambda(g, \vartheta; \pi, \tau, \rho). \tag{4}$$

Independent variation with respect to the potentials (g, ϑ, ω) and their conjugate momenta (π, τ, ρ) yields

$$\delta\mathcal{L} = \delta g_{\mu\nu} \frac{\delta\mathcal{L}}{\delta g_{\mu\nu}} + \delta\vartheta^\alpha \wedge \frac{\delta\mathcal{L}}{\delta\vartheta^\alpha} + \delta\omega^\alpha{}_\beta \wedge \frac{\delta\mathcal{L}}{\delta\omega^\alpha{}_\beta} + \frac{\delta\mathcal{L}}{\delta\pi^{\mu\nu}} \wedge \delta\pi^{\mu\nu} + \frac{\delta\mathcal{L}}{\delta\tau_\alpha} \wedge \delta\tau_\alpha$$

$$+ \frac{\delta\mathcal{L}}{\delta\rho_\alpha{}^\beta} \wedge \delta\rho_\alpha{}^\beta + d(\delta g_{\mu\nu}\pi^{\mu\nu} + \delta\vartheta^\alpha \wedge \tau_\alpha + \delta\omega^\alpha{}_\beta \wedge \rho_\alpha{}^\beta), \tag{5}$$

which implicitly defines the *first order* equations (their explicit form will not be needed).

The Hamiltonian formulation displays the "time" evolution of the physical system. To proceed in a covariant way we consider instants of "time" to be given by spacelike hypersurfaces determined by a function t. A "time evolution" vector field N displaces one constant t space-like hypersurface Σ_t to another; consequently $i_N dt = 1$. Time evolution is then given by the Lie derivative where $\pounds_N = i_N d + d i_N$ on the components of form fields. We regard t and N as being fixed when we vary our physical variables. One way to obtain a set of Hamiltonian equations is the "spatial restriction" (pullback) and "time projection" (evaluation on the vector field N) of the first order equations. This would give the initial value constraint and dynamical evolution equations respectively.

The same equations can also be obtained from a Hamiltonian. To obtain the Hamiltonian in a covariant form we decompose the Lagrangian as follows:

$$\mathcal{L} \equiv dt \wedge i_N \mathcal{L}$$
$$= dt \wedge (\pounds_N g_{\mu\nu}\pi^{\mu\nu} + \pounds_N \vartheta^\alpha \wedge \tau_\alpha + \pounds_N \omega^\alpha{}_\beta \wedge \rho_\alpha{}^\beta - \mathcal{H}(N)). \tag{6}$$

In this way we find the *Hamiltonian* 3-form $\mathcal{H}(N) = N^\mu \mathcal{H}_\mu + d\mathcal{B}(N)$, where

$$N^\mu \mathcal{H}_\mu := i_N \Lambda + Dg_{\mu\nu} \wedge i_N \pi^{\mu\nu} - \Theta^\alpha \wedge i_N \tau_\alpha - \Omega^\alpha{}_\beta \wedge i_N \rho_\alpha{}^\beta - i_N \vartheta^\alpha \wedge D\tau_\alpha$$
$$- \; i_N \omega^\alpha{}_\beta (D\rho_\alpha{}^\beta - g_{\alpha\nu}\pi^{\beta\nu} - g_{\mu\alpha}\pi^{\mu\beta} + \vartheta^\beta \wedge \tau_\alpha), \tag{7}$$
$$\mathcal{B}(N) := i_N \vartheta^\alpha \tau_\alpha + i_N \omega^\alpha{}_\beta \rho_\alpha{}^\beta. \tag{8}$$

The Hamiltonian 3-form $\mathcal{H}(N)$ is the same quantity which arises when one considers Noether's second theorem for a local translation along a vector field N applied to the first order Lagrangian (4), as we now briefly explain.

Geometric theories are invariant under local diffeomorphisms. For a first order Lagrangian of the form $\mathcal{L} = d\varphi \wedge p - \Lambda$ the general variational formula

$$\delta\mathcal{L} = d(\delta\varphi \wedge p) + \delta\varphi \wedge \frac{\delta\mathcal{L}}{\delta\varphi} + \frac{\delta\mathcal{L}}{\delta p} \wedge \delta p, \tag{9}$$

must be identically satisfied if the variations are generated by an infinitesimal diffeomorphism. Hence

$$\pounds_N \mathcal{L} = di_N \mathcal{L} \equiv d(\pounds_N \varphi \wedge p) + \pounds_N \varphi \wedge \frac{\delta\mathcal{L}}{\delta\varphi} + \frac{\delta\mathcal{L}}{\delta p} \wedge \pounds_N p. \tag{10}$$

Consequently the "Hamiltonian 3-form"

$$\mathcal{H}(N) := \pounds_N \varphi \wedge p - i_N \mathcal{L}, \tag{11}$$

satisfies a differential identity

$$d\mathcal{H}(N) \equiv (\text{terms proportional to field equations}). \tag{12}$$

In addition to this Noether differential identity, because displacement along N is a local symmetry, we also have an algebraic identity. Using $\mathcal{L} = d\varphi \wedge p - \Lambda$ in (11) we rearrange the Hamiltonian into the form $\mathcal{H}(N) = N^\mu \mathcal{H}_\mu + d\mathcal{B}(N)$ and then substitute $d\mathcal{H}(N) = d(N^\mu \mathcal{H}_\mu) = dN^\mu \wedge \mathcal{H}_\mu + N^\mu d\mathcal{H}_\mu$ into the differential identity (12). The coefficient of dN^μ gives an algebraic identity,

$$\mathcal{H}_\mu \equiv (\text{terms proportional to field equations}). \tag{13}$$

From (12) we see that "on shell" (i.e., when the field equations are satisfied) the Hamiltonian 3-form is a "conserved current" and the value of the Hamiltonian $H(N)$, the integral of the $\mathcal{H}(N)$ over a space-like hypersurface Σ, is a conserved quantity. From the algebraic identity (13), $\mathcal{H}_\mu$ vanishes when evaluated for solutions of the field

equations. Thus, "on shell", the value of Hamiltonian $H(N)$ depends only on the term $d\mathcal{B}(N)$. By the generalized Stokes theorem, the value of the Hamiltonian is just the integral of the 2-form $\mathcal{B}(N)$ over the 2-surface boundary $\partial\Sigma$ of the 3-hypersurface Σ. This value should determine the *energy* and the other quasilocal quantities. However, just as with other Noether currents, $\mathcal{H}(N)$ is not unique; we can add to it a total differential without affecting the conservation relation (12), this amounts to modifying $\mathcal{B}(N)$ (8). Such a modification does not affect the Hamiltonian equations of motion. The Hamiltonian field equations are obtained by varying the action (6). The variation of the above Hamiltonian has the form

$$\delta\mathcal{H}(N) = \text{(field equation terms)} + d\mathcal{C}(N), \tag{14}$$

where

$$\mathcal{C}(N) := i_N(\delta g_{\mu\nu}\pi^{\mu\nu} + \delta\vartheta^\alpha \wedge \tau_\alpha + \delta\omega^\alpha{}_\beta \wedge \rho_\alpha{}^\beta). \tag{15}$$

A modification of $\mathcal{B}(N)$ (8) affects only the total derivative term $d\mathcal{C}(N)$ which, upon integration over Σ becomes a boundary term.

3. Expressions for quasilocal quantities

Our purpose is to find quasilocal quantities for gravity theories. Hence we want to determine the expression for the term $\mathcal{B}(N)$ in the Hamiltonian. From the variational principle we know that a total differential term can be added to the Hamiltonian without changing the field equations. But this, of course, will change the expression of $\mathcal{B}(N)$, and thereby the value of the quasilocal quantities. What is the proper "physical" expression for quasilocal quantities? We have already noted that the requirement of having the desired limiting values is far too weak. We use certain additional theoretical criteria involving the boundary term in the variation of the Hamiltonian to severely restrict the form of a "good" expression for the term $\mathcal{B}(N)$.

3.1. Requirements

Consider the total differential term $\mathcal{C}$ in the variation of the Hamiltonian. We propose three theoretical requirements on the term $\mathcal{C}$ in order to limit the quasilocal boundary expression $\mathcal{B}$:

- **[R1] Well-defined Requirement**
  - *The term $\mathcal{C}$ must vanish at the boundary in general both for finite and infinite regions.*

- **[R2] Symplectic Structure Requirement**
  - *The term $\mathcal{C}$ must reflect the symplectic structure [35], which tells us which physical variables are "control" variables (i.e., we control them at the*

boundary), and which are "response" variables (i.e., are determined by physical laws).

- **[R3] Covariant Requirement**
  - *The control and response variables must each appear in the form of a covariant combination* (as befits a covariant theory).

With respect to the Well-defined Requirement R1, the expression of the term $\mathcal{C}(N)$ in equation (15) is no problem for a finite boundary. We need merely fix the appropriate variables to vanish at the boundary. For a boundary at infinity, however, one needs to consider a limit and the natural rate of fall off in the asymptotic region. In this case, the term $\mathcal{C}(N)$ in equation (15) is nonvanishing generally. (Specific examples are the Einstein theory [22] and the Poincaré gauge theory (PGT) for solutions which have asymptotically zero or constant curvature.) In order to correct $\mathcal{C}(N)$ to satisfy R1, one *must* modify the term $\mathcal{B}$ in equation (8).

3.2. Well-defined version

Some time ago within the context of metric compatible theories, i.e., the PGT, one of us [20] proposed a modified version of the boundary term (8):

$$\mathcal{B}(N) = i_N\vartheta^\alpha\Delta\tau_\alpha + i_N\omega^\alpha{}_\beta\,\Delta\rho_\alpha{}^\beta + \Delta\omega^\alpha{}_\beta \wedge i_N\rho_\alpha{}^\beta. \tag{16}$$

With this adjustment the total differential term $\mathcal{C}(N)$ in the variation of $\mathcal{H}(N)$ takes the form

$$\mathcal{C}(N) = \delta i_N\vartheta^\alpha\Delta\tau_\alpha - \delta\vartheta^\alpha \wedge i_N\tau_\alpha + \delta i_N\omega^\alpha{}_\beta\,\Delta\rho_\alpha{}^\beta + \Delta\omega^\alpha{}_\beta \wedge \delta i_N\rho_\alpha{}^\beta, \tag{17}$$

where, for any quantity α, $\mathring{\alpha}$ is the *reference configuration* value of α which is fixed under variation, and $\Delta\alpha := \alpha - \mathring{\alpha}$. This expression should vanish at spatial infinity, in particular for asymptotically flat or constant curvature PGT solutions. But for some theories, e.g., the PGT with a massless torsion field [43], the boundary term $\mathcal{B}(N)$ still needs adjustment. Hecht recognized this and proposed an important modification [24] to the $i_N\omega$ factor. (We have already noted that the Hecht expression has since been successfully tested at spatial *and* null infinity for both asymptotically flat *and* asymptotically constant (negative) curvature solutions [25, 26].)

However we can also add a term $\Delta\vartheta^\alpha \wedge i_N\tau_\alpha$ into the original expression (16) to give the formula more symmetry. Then

$$\mathcal{B}'(N) = i_N\vartheta^\alpha\Delta\tau_\alpha + \Delta\vartheta^\alpha \wedge i_N\tau_\alpha + i_N\omega^\alpha{}_\beta\,\Delta\rho_\alpha{}^\beta + \Delta\omega^\alpha{}_\beta \wedge i_N\rho_\alpha{}^\beta, \tag{18}$$

and

$$\mathcal{C}'(N) = \delta i_N\vartheta^\alpha\Delta\tau_\alpha + \Delta\vartheta^\alpha \wedge \delta i_N\tau_\alpha + \delta i_N\omega^\alpha{}_\beta\,\Delta\rho_\alpha{}^\beta + \Delta\omega^\alpha{}_\beta \wedge \delta i_N\rho_\alpha{}^\beta. \tag{19}$$

From these considerations we see that the Well-Defined requirement R1 cannot determine the quasilocal term $\mathcal{B}(N)$ uniquely. Moreover neither $\mathcal{C}(N)$ nor $\mathcal{C}'(N)$ satisfy the Covariant Requirement R3.

3.3. Covariant symplectic structure version

More recently we have improved these expressions and obtained several different boundary terms, all of them satisfy the requirements R1, R2 and R3. They include, for each independent k-form field φ (e.g., the connection, the coframe, the metric), one of two possible covariant boundary terms:

$$\mathcal{B}_\varphi(N) = i_N\varphi \wedge \Delta p - (-1)^k \Delta\varphi \wedge i_N\overset{\circ}{p} \quad \text{or} \quad \mathcal{B}_p(N) = i_N\overset{\circ}{\varphi} \wedge \Delta p - (-1)^k \Delta\varphi \wedge i_N p, \tag{20}$$

depending upon whether φ or its conjugate momenta p is held fixed (controlled) on the boundary. Then the variation of the Hamiltonian, in addition to the field equations, includes a boundary term which is a projection onto the boundary of a covariant *symplectic* structure:

$$d i_N(\delta\varphi \wedge \Delta p) \qquad \text{or} \qquad d i_N(-\Delta\varphi \wedge \delta p), \tag{21}$$

respectively, which reflects the choice of control mode.

Thus for the geometric fields we take

$$\mathcal{B}(N) = \left\{ \begin{array}{c} -\Delta g_{\mu\nu}\, i_N\overset{\circ}{\pi}^{\mu\nu} \\ -\Delta g_{\mu\nu}\, i_N\pi^{\mu\nu} \end{array} \right\} + \left\{ \begin{array}{c} i_N\vartheta^\alpha \Delta\tau_\alpha + \Delta\vartheta^\alpha \wedge i_N\overset{\circ}{\tau}_\alpha \\ i_N\overset{\circ}{\vartheta}^\alpha \Delta\tau_\alpha + \Delta\vartheta^\alpha \wedge i_N\tau_\alpha \end{array} \right\}$$
$$+ \left\{ \begin{array}{c} i_N\omega^\alpha{}_\beta \Delta\rho_\alpha{}^\beta + \Delta\omega^\alpha{}_\beta \wedge i_N\overset{\circ}{\rho}_\alpha{}^\beta \\ i_N\overset{\circ}{\omega}^\alpha{}_\beta \Delta\rho_\alpha{}^\beta + \Delta\omega^\alpha{}_\beta \wedge i_N\rho_\alpha{}^\beta \end{array} \right\}, \tag{22}$$

where the upper (lower) line in each bracket is to be selected if the field (momentum) is controlled. Hence, as in thermodynamics, there are several kinds of "energy", each corresponds to the work done in a different (ideal) physical process [35], [8].

For the geometric variables, the total differential term in $\delta\mathcal{H}(N)$ is of the boundary projection form $d i_N\mathcal{C}$ with $\mathcal{C}$ now given by the covariant expression

$$\mathcal{C} = \left\{ \begin{array}{c} \delta g_{\mu\nu}\, \Delta\pi^{\mu\nu} \\ -\Delta g_{\mu\nu}\, \delta\pi^{\mu\nu} \end{array} \right\} + \left\{ \begin{array}{c} \delta\vartheta^\alpha \wedge \Delta\tau_\alpha \\ -\Delta\vartheta^\alpha \wedge \delta\tau_\alpha \end{array} \right\} + \left\{ \begin{array}{c} \delta\omega^\alpha{}_\beta \wedge \Delta\rho_\alpha{}^\beta \\ -\Delta\omega^\alpha{}_\beta \wedge \delta\rho_\alpha{}^\beta \end{array} \right\}, \tag{23}$$

where again the upper (lower) line in each bracket corresponds to controlling the field (momentum).

Of course we cannot expect to get completely covariant expressions when we have a non-covariant dynamic variable such as a connection. Note however that $\Delta\omega$, being the difference between two connections, is tensorial, so our quasilocal boundary expressions are *covariant*—aside from the manifestly non-covariant explicit connection terms in $\mathcal{B}$. These $i_N\omega$ terms include a real covariant physical effect plus an unphysical

(noncovariant) dynamical reference frame effect. These two effects can be separated by using the identity

$$(i_N \omega^\alpha{}_\beta)\vartheta^\beta \equiv i_N \Theta^\alpha + DN^\alpha - \pounds_N \vartheta^\alpha, \tag{24}$$

to replace the $i_N \omega$ factor within the last bracket in (22) with two covariant terms plus a manifestly non-covariant term of the form $(\pounds_N \vartheta^\alpha)_\beta \Delta \rho_\alpha{}^\beta$. This $\pounds_N \vartheta \Delta \rho$ piece is really needed in our general Hamiltonian to generate the dynamics due to the gauge freedom of the reference frame — but it represents an unphysical (observer dependent) contribution to the quasilocal quantities. For the purposes of calculating the physical value of the quasilocal quantities it should be dropped. (Note that if N is a Killing vector then this term can be made to vanish for a suitable choice of frame.)

3.4.   Uniqueness

An important property of our expressions is *uniqueness*. Of course, the uniqueness property depends on the proposed requirements. Under our three requirements (R1, R2 and R3), our expressions are the only possible ones.

According to our covariant symplectic structure requirement (R2 + R3), for each field potential φ and its momenta p, the boundary term of the variation of the Hamiltonian must be one of the following two types

$$di_N(\delta\varphi \wedge \Delta p) \qquad \text{or} \qquad di_N(-\Delta\varphi \wedge \delta p). \tag{25}$$

The only acceptable modification to the Hamiltonian which preserves the covariant symplectic structure requirement is of the form $di_N \mathcal{F}$, a projection of a 4-covariant form onto the boundary. The 3-form $\mathcal{F}$ must depend on the control variables only algebraically † and must be scalar-valued. Working with our dynamic variable pairs $(g_{\mu\nu}, \pi^{\mu\nu})$, $(\theta^\alpha, \tau_\alpha)$ and $(\omega^\alpha{}_\beta, \rho_\alpha{}^\beta)$, the only acceptable combination is of the form $di_N(\Delta\varphi \wedge \Delta p)$. This just switches between the two control modes:

$$\mathcal{B}_p = \mathcal{B}_\varphi - i_N(\Delta\varphi \wedge \Delta p). \tag{26}$$

Of course one could add some covariant quantity which doesn't depend on the dynamic variable. That would leave the Hamiltonian variation symplectic structure unchanged but would "renormalize" the quasilocal quantities—albeit in an "unphysical way". We rule this out by normalizing our quasilocal expressions so that they vanish if the dynamic variables equal the reference values on the boundary.

Although we have determined the expressions 'uniquely', there still exist two undetermined things. One is the displacement vector field N. For example, if we

† Since if it depends on the differential of the control variable, we must fix the differential of this variable at the boundary. This is forbidden by the symplectic structure.

want to calculate the energy of a physical system in a finite region, we have not yet specified how to select the appropriate timelike vector N. The other is the reference configuration. We will look closer at these issues later.

3.5. Spacetime version

Now let's go back to the Lagrangian level using the same requirements in order to get a completely covariant analysis. In equation (5), the boundary term satisfies the Covariant Symplectic Structure Requirement R2 and R3, but not the Well-defined Requirement R1.

For the purpose of getting boundary terms satisfying our requirements, we considered ways to modify the Lagrangian by adding an extra total differential term. In a first order Lagrangian we found that we could include a boundary term by letting

$$\mathcal{L} := d\varphi \wedge p - \Lambda - d\mathcal{K}, \tag{27}$$

where $\mathcal{K} = \Delta\varphi \wedge \mathring{p}$ or $\mathcal{K} = \Delta\varphi \wedge p$. Then

$$\delta\mathcal{L} = (\text{field equations}) + d\left(\left\{ \begin{matrix} \delta\varphi \wedge \Delta p \\ -\Delta\varphi \wedge \delta p \end{matrix} \right\} \right). \tag{28}$$

We can relate this to the Hamiltonian analysis for

$$\delta i_N \mathcal{L} \equiv i_N \delta\mathcal{L} = i_N (\text{field equations}) + i_N d\left(\left\{ \begin{matrix} \delta\varphi \wedge \Delta p \\ -\Delta\varphi \wedge \delta p \end{matrix} \right\} \right). \tag{29}$$

The latter term can be rearranged into

$$\pounds_N \left(\left\{ \begin{matrix} \delta\varphi \wedge \Delta p \\ -\Delta\varphi \wedge \delta p \end{matrix} \right\} \right) - d i_N \left(\left\{ \begin{matrix} \delta\varphi \wedge \Delta p \\ -\Delta\varphi \wedge \delta p \end{matrix} \right\} \right), \tag{30}$$

which consists of a total time derivative (which integrates into a term at the initial and final times and constitutes a canonical transformation) and a spatial boundary term which is identical to our covariant symplectic Hamiltonian boundary variation term. By the way, it is here that one can see the covariant origin of our required form for the projection of a covariant quantity (21).

4. Black hole thermodynamics

Here we will use arguments similar to those of Brown and York [44, 45] and of Wald [46] (see also [47]) along with our symplectic Hamiltonian expressions to derive the first law of black hole thermodynamics for these general geometric gravity theories.

For the theories of gravity that we are considering, the null rays and causal structure are governed only by the metric and its associated Riemannian geometry. Hence the basic geometry of black holes in these theories should be the same as that of GR.

The quantity corresponding to the temperature in black hole thermodynamics is the *surface gravity*, κ, defined at any point of a Killing horizon by [48]

$$\chi^\alpha \nabla_\alpha \chi^\beta =: \kappa \chi^\beta, \tag{31}$$

where χ^α is the Killing vector field normal to the Killing horizon and ∇ is the Riemannian covariant derivative. Hawking proved that the event horizon of a stationary black hole is a Killing horizon in general relativity [49]. For certain general gravity theories it has been shown that if a black hole has a bifurcate Killing horizon (the Killing vector vanishes at this horizon) then the constancy of the surface gravity holds [50]. For the theories under consideration here we presume that the black hole has a bifurcate Killing horizon and that the zeroth law is satisfied: the surface gravity (which is proportional to the temperature) is constant over the entire horizon [51].

The first law of thermodynamics concerns the conservation of energy. Because our Hamiltonians satisfy a conservation law, the corresponding formula for gravitating systems, in particular black holes, can be obtained and the quantity corresponding to the entropy identified.

The basic idea is to consider the region between the horizon and infinity. We wish to vary certain parameters such as the total energy and angular momentum. However in the formulations discussed earlier such quantities were not control parameters. To obtain the necessary additional free variations on the boundary we must allow the displacement vector field to vary. Then the variation of the Hamiltonian 3-form, including the variation of the displacement vector field, is of the form

$$\begin{aligned}
\delta\mathcal{H}(N) &= (\text{field equation terms}) + (\delta N)^\mu \mathcal{H}_\mu \\
&\quad + d(i_N(\text{symplectic control mode terms}) + \mathcal{B}(\delta N)),
\end{aligned} \tag{32}$$

with the $\mathcal{H}_\mu$ term vanishing by the initial value constraints. Note that the response variable for the displacement vector is the "component" of the quasilocal quantities. Here $\mathcal{B}$ can be any one of our boundary terms representing any control mode as long as the "reference frame energy" is removed by dropping the $\pounds_N \vartheta$ contribution via the $i_N\omega$ replacement (24). Following Brown and York [44], we now introduce the *microcanonical* Hamiltonian via a Legendre transformation:

$$\mathcal{H}_{\text{micro}}(N) = \mathcal{H}(N) - d\mathcal{B}(N). \tag{33}$$

The variation of the microcanonical action is of the form

$$\begin{aligned}
\delta(S_{\text{micro}}[N]) &= \int_\Sigma (\text{field equation terms}) - (\delta N)^\mu \mathcal{H}_\mu \\
&\quad - \oint_{\partial\Sigma} (i_N(\text{symplectic control mode terms}) + (\delta\mathcal{B})(N)).
\end{aligned} \tag{34}$$

The variation of S_{micro} should vanish on a solution with the control variables fixed at the boundary. Suppose the hypersurface includes a boundary with two components,

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one at infinity and the other at the bifurcate Killing horizon, H. Then we get a kind of energy-momentum conservation law:

$$0 = \oint_{\partial\Sigma} (\delta\mathcal{B})(N) = \oint_{\infty} (\delta\mathcal{B})(N) - \oint_{H} (\delta\mathcal{B})(N). \tag{35}$$

Now we shall choose our variations to be such that they perturb the asymptotic values of the total quantities.

Let X_t and X_φ denote the Killing vector fields which approach the time translation and rotation at the infinity. Then the total energy is the value of $H(X_t) = \oint_\infty \mathcal{B}(X_t)$ and the total angular momentum is the value of $H(X_\varphi) = \oint_\infty \mathcal{B}(X_\varphi)$. It is worth remarking that this is true for all the possible control modes, asymptotically they give the same total results for they differ by terms with more rapid fall off.

We assume as usual that the Killing vector χ can be decomposed into

$$\chi = X_t + \Omega_H X_\varphi, \tag{36}$$

and vanishes on the bifurcate Killing horizon, where Ω_H is the "angular velocity of the horizon".

Now let the displacement N be the Killing vector χ. Then

$$\oint_\infty (\delta\mathcal{B})(N) = \delta E + \Omega_H\, \delta J. \tag{37}$$

On the other hand, for the surface integration over the horizon boundary, because the Killing field vanishes on the horizon the only terms that can contribute involve $\overset{\circ}{D}\chi$ or $D\chi = \overset{\circ}{D}\chi + \Delta\omega\chi$, both of which which reduce to $\nabla\chi$ on the horizon (since the affine connections differ from the Levi-Civita connection by tensor terms multiplied by the vector field which vanishes on H). Hence we get

$$\oint_H (\delta\mathcal{B})(\chi) = \oint_H \nabla_\beta\chi^\alpha \delta\, \rho_\alpha{}^\beta = \kappa \oint_H \epsilon_\beta{}^\alpha\, \delta\rho_\alpha{}^\beta =: \kappa\, \delta S, \tag{38}$$

here we have assumed that $\nabla_\alpha\chi_\beta = \kappa\epsilon_{\alpha\beta}$, where $\epsilon_{\alpha\beta}$ is the bi-normal to the horizon. This has the form of the first law and identifies an expression for the *black hole entropy S* for our general gravity theories to be dependent on the horizon components of $\rho_\alpha{}^\beta$, which is the field momentum 2-form conjugate to the curvature. This result is consistent with the well known Einstein theory value, where the integral of $\epsilon^{\alpha\beta}\rho_{\alpha\beta}$ is proportional to the area. Remarkably, it is of the same form as an expression found previously for other general gravity theories which were based on Riemannian geometry with no torsion or non-metricity but, on the other hand, were allowed to contained arbitrarily high derivatives of the curvature [51, 46, 45]. These works used various techniques ranging from the Noether-Charge analysis of Iyer and Wald to the microcanonical functional integral method of Brown. The extent to which these results can be combined with our work is not yet clear to us.

5. Some theories of gravity

Our general formalism readily specializes to either coordinate or orthonormal frames, moreover it is suitable for the most general metric-affine gravity theories [42, 52]. (For earlier proposals for quasilocal quantities for such theories see §5.7 in [42].) Nevertheless, it also readily specializes to less general geometries like Riemann-Cartan, Riemannian or teleparallel and to gravity theories formulated in such geometries.

Teleparallel theories (i.e., curvature vanishes), see, e.g., [53, 54], are somewhat special; hence we leave the treatment of their covariant Hamiltonian formalism, and quasilocal quantities for future work. For the Poincaré gauge theory (PGT) [55, 56] one could impose the metric compatible connection condition via a Lagrange multiplier. However, a neater method is to use orthonormal frames and restrict the connection algebraically: $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$. Then one need merely drop all the g, π terms from our expressions. The most general potential $\Lambda(\vartheta, \tau, \rho)$ contains 3 independent quadratic terms in τ and 6 independent quadratic terms in ρ:

$$\tau_\alpha = -\frac{1}{\chi} * (\sum_{n=1}^{3} a_n{}^{(n)}\Theta_\alpha), \qquad \rho^{\alpha\beta} = -\frac{a_0}{2\chi}\eta^{\alpha\beta} - \frac{1}{\kappa} * (\sum_{n=1}^{6} b_n{}^{(n)}\Omega^{\alpha\beta}), \quad (39)$$

which express the momenta linearly in terms of the algebraically irreducible parts of the torsion and curvature. Here we have introduced the convenient notation $\eta^{\alpha\cdots} := *(\vartheta^\alpha \wedge \cdots)$, with $\eta := *1 = \vartheta^0 \wedge \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3$.

The Einstein-Cartan theory is a special case of the PGT with the Lagrangian

$$\mathcal{L}_{EC} = \frac{1}{2\chi}\Omega^\alpha{}_\beta \wedge \eta_\alpha{}^\beta + \frac{\Lambda_{cos}}{\chi}\eta, \qquad (40)$$

(where Λ_{cos} is the cosmological constant and $\chi := 8\pi G/c^4 = 8\pi$ in geometric units). The variables are the orthonormal frame and the metric compatible (antisymmetric) connection one-form. The associated conjugate momenta are

$$\tau_\alpha = 0, \qquad \rho_\alpha{}^\beta = \frac{1}{2\chi}\eta_\alpha{}^\beta. \qquad (41)$$

Properly we should introduce these momenta and regard (41) as constraints which can be enforced with Lagrange multipliers. However, because these "momenta constraints" are purely algebraic, we can take a short cut and directly use (40) as our first order Lagrangian. The corresponding quasilocal expressions are

$$\mathcal{B}_\omega = \frac{1}{2\chi}(\Delta\omega^\alpha{}_\beta \wedge i_N\mathring{\eta}_\alpha{}^\beta + i_N\omega^\alpha{}_\beta\,\Delta\eta_\alpha{}^\beta), \qquad (42)$$

$$\mathcal{B}_\vartheta = \frac{1}{2\chi}(\Delta\omega^\alpha{}_\beta \wedge i_N\eta_\alpha{}^\beta + i_N\mathring{\omega}^\alpha{}_\beta\,\Delta\eta_\alpha{}^\beta). \qquad (43)$$

The gravity theory which attracts the most interest is Einstein's General Relativity. The Riemannian geometry of GR does not have an independent connection while our

formalism is geared to independent variations of the frame, metric and connection, Yet there is no incompatibility, indeed one can proceed in many ways including: (1) impose the metric compatible and vanishing torsion conditions via Lagrange multipliers, (2) use coordinate frames, algebraically impose the symmetry of the connection coefficients and impose metric compatibility via a Lagrange multiplier, (3) use coordinate frames, algebraically impose the symmetry of the connection coefficients and (for vacuum and all non-derivative coupled sources) obtain the metric compatibility condition as a field equation via a Palatini type variation, (4) use the orthonormal frame EC theory and impose the vanishing torsion condition via a Lagrange multiplier, (5) use the orthonormal frame EC theory and obtain (for vacuum and all non-derivative coupled sources) the vanishing torsion condition as a field equation.

The latter method is technically the simplest and we will use it here to obtain our quasilocal quantities for GR. All of the other procedures give similar results. Similar but not necessarily identical—because the boundary control differs—for example $\Delta\omega^{\alpha}{}_{\beta} \wedge \delta\eta_{\alpha}{}^{\beta} = \Delta\omega^{\alpha}{}_{\beta} \wedge \delta\vartheta^{\mu} \wedge \eta_{\alpha}{}^{\beta}{}_{\mu}$ in terms of orthonormal (co)frames, but it equals $\Delta\Gamma^{\alpha}{}_{\beta\gamma} \delta(\sqrt{-g}g^{\beta\sigma}) \delta^{\nu\gamma}_{\alpha\sigma} d^3x_{\nu}$ in a coordinate basis. A comprehensive investigation covering all these options is underway.

6. Quasilocal values in general relativity

We now consider our quasilocal expressions for general relativity. This will help to give a better understanding of their mathematical nature and physical meaning and will permit a comparison with the results obtained by other authors.

The Hilbert Lagrangian (40) of the Einstein(-Cartan) theory (dropping the cosmological constant and often the constant 2χ for simplicity) can be spacetime decomposed according to

$$
\begin{aligned}
\mathcal{L} &= dt \wedge i_N \mathcal{L} \\
&= dt \wedge [\pounds_N \omega^{\alpha}{}_{\beta} \wedge \eta_{\alpha}{}^{\beta} + N^{\mu}\Omega^{\alpha}{}_{\beta} \wedge \eta_{\alpha}{}^{\beta}{}_{\mu} + i_N\omega^{\alpha}{}_{\beta} D\eta_{\alpha}{}^{\beta} - d(i_N\omega^{\alpha}{}_{\beta}\eta_{\alpha}{}^{\beta})] \,,
\end{aligned}
\tag{44}
$$

which identifies the Hamiltonian 3-form as

$$
\mathcal{H} = -N^{\mu}\Omega^{\alpha}{}_{\beta} \wedge \eta_{\alpha}{}^{\beta}{}_{\mu} - i_N\omega^{\alpha}{}_{\beta} D\eta_{\alpha}{}^{\beta} + d(i_N\omega^{\alpha}{}_{\beta}\eta_{\alpha}{}^{\beta}) \,.
\tag{45}
$$

On a constant $t = x^0$ surface, the principal term in this Hamiltonian, $-N^{\mu}\Omega^{\alpha}{}_{\beta}\wedge\eta_{\alpha}{}^{\beta}{}_{\mu} = N^{\mu}2G^{\nu}{}_{\mu}\eta_{\nu}$, just reduces to the familiar ADM value $2N^{\mu}G^0{}_{\mu}$ (the $i_N\omega$ rotation gauge generator term is proportional to the torsion). However the boundary term still needs adjustment [22]. We propose using one of the Hamiltonian boundary terms (42, 43). Removing the observers "reference frame energy-momentum" $\sim \pounds_N\vartheta\Delta\eta$ via the identity (24) gives our covariant quasilocal boundary expressions for GR:

$$
\mathcal{B}_{\omega} = \Delta\omega^{\alpha}{}_{\beta} \wedge i_N\mathring{\eta}_{\alpha}{}^{\beta} + D_{\beta}N^{\alpha} \Delta\eta_{\alpha}{}^{\beta} \,,
\tag{46}
$$

$$\mathcal{B}_\vartheta = \Delta\omega^\alpha{}_\beta \wedge i_N \eta_\alpha{}^\beta + \overset{\circ}{D}_\beta \overset{\circ}{N}^\alpha \, \Delta\eta_\alpha{}^\beta \,. \tag{47}$$

For the simple alternative choice of a coordinate basis with the metric and symmetric connection as the variables, the derivation of the Hamiltonian and its boundary terms is formally the same. The control variables are then $\sqrt{-g}g^{\beta\sigma}$ and $\Gamma^\alpha{}_{\beta\lambda}$ respectively, and the components of the quasilocal expressions (46, 47) take (with $\mathcal{B} := (1/4)\mathcal{B}^{\tau\rho}\epsilon_{\tau\rho\mu\nu}dx^\mu \wedge dx^\nu$) the form

$$\mathcal{B}_g^{\tau\rho} = N^\mu(-g)^{\frac{1}{2}} g^{\beta\sigma} \, \Delta\Gamma^\alpha{}_{\beta\lambda} \, \delta^{\tau\rho\lambda}_{\alpha\sigma\mu} + \overset{\circ}{D}_\beta \overset{\circ}{N}^\alpha \, \Delta(\sqrt{-g}g^{\beta\sigma}) \, \delta^{\tau\rho}_{\alpha\sigma} \,, \tag{48}$$

$$\mathcal{B}_\Gamma^{\tau\rho} = N^\mu(-\overset{\circ}{g})^{\frac{1}{2}} \overset{\circ}{g}^{\beta\sigma} \, \Delta\Gamma^\alpha{}_{\beta\lambda} \, \delta^{\tau\rho\lambda}_{\alpha\sigma\mu} + D_\beta N^\alpha \, \Delta(\sqrt{-g}g^{\beta\sigma}) \, \delta^{\tau\rho}_{\alpha\sigma} \,. \tag{49}$$

These new quasilocal boundary expressions are similar to certain results obtained by earlier investigators. With a unit displacement, the first term of $\mathcal{B}_g$ by itself,

$$U_\mu{}^{\tau\rho}(g, \Delta\Gamma) := (-g)^{\frac{1}{2}} g^{\beta\sigma} \, \Delta\Gamma^\alpha{}_{\beta\lambda} \, \delta^{\tau\rho\lambda}_{\alpha\sigma\mu} \,, \tag{50}$$

if we use a vanishing reference connection, reduces to the *Freud superpotential* [57]:

$$U_\mu{}^{\tau\rho}(g, \Gamma) \equiv (-g)^{-\frac{1}{2}} g_{\mu\nu}\partial_\lambda[-g(g^{\tau\nu}g^{\rho\lambda} - g^{\rho\nu}g^{\tau\lambda})] \,. \tag{51}$$

Using just this superpotential as the Hamiltonian boundary term is equivalent to integrating its divergence, the *Einstein pseudotensor*, over the spacelike hypersurface. Our second term, without the Δ, has the form $2D^{[\rho}N^{\tau]}$, which is the well known Komar expression [58]. Several investigators, working from different perspectives, have arrived at an expression which is the sum of these two terms:

$$N^\mu U_\mu{}^{\tau\rho}(g, \Delta\Gamma) + 2\sqrt{-g}\, g^{\mu[\tau}\delta^{\rho]}_\lambda \overset{\circ}{D}_\mu N^\lambda \equiv 2\sqrt{-g}\, N^{[\tau}K^{\rho]} + 2\sqrt{-g}\, D^{[\rho}N^{\tau]}, \tag{52}$$

where $K^\rho := 2g^{\beta[\lambda}\Delta\Gamma^{\rho]}{}_{\beta\lambda}$. This was obtained by Chruściel [59] and also by Katz [7] using a background metric while Sorkin [60] used only a background connection. Chruściel's analysis is especially relevant to us since he also used the Kijowski-Tulczyjew symplectic-Hamiltonian ideas in much the same spirit as we have done; his work even includes a second expression corresponding to the conjugate control mode. Most of these earlier investigators recognized the quasilocal possibilities of their results, but they were primarily concerned with obtaining and discussing the virtues of this expression for total energy-momentum and angular momentum along with reconciling the Komar expression with those obtained by other techniques. Mention should also be made of an interesting covariant investigation of Ferraris and Francaviglia [61] which uses a global background symmetric connection and has also arrived at the same expression.

Although the expression (52) obtained by the aforementioned investigators is quite close to our (48), an important difference lies in the presence of the Δ in the "Komar" term. In most practical quasilocal calculations this amounts to a 'difference which makes no difference'. However we believe that this Δ is really needed for the most

general considerations (in particular it is necessary to get the asymptotically correct behavior for the boundary term in the Hamiltonian variation). There are precedents for this factor beginning with the work of Beig and Ó Murchadha [38].

To see this it is necessary to do a space time split of our expressions. We '3+1' decompose expression (47) using an orthonormal frame with one leg $e_\perp$ normal to the hypersurface. The extrinsic curvature of the spatial hypersurface is related to the spatial restriction of certain 4-dimensional connection one-form components:

$$\omega^\perp{}_j = -K_{jl}\vartheta^l \,, \tag{53}$$

while the spatial restriction of the spatial components $\omega^i{}_j$ of the 4-dimensional connection one-form is just the 3-dimensional connection one-form (for details of the technique see [62]). In this way we obtain

$$\mathcal{B}_\vartheta = \Delta\omega^{ij} \wedge N^\perp \eta_{ij\perp} + 2\Delta\omega^{\perp j} \wedge N^k \eta_{\perp jk} + (\mathring{D}^j \mathring{N}^\perp - \mathring{D}^\perp \mathring{N}^j)\,\Delta\eta_{\perp j}\,, \tag{54}$$

where we have dropped certain terms, including $\mathring{D}^i \mathring{N}^j \,\Delta\eta_{ij}$, since they are proportional to $\vartheta^\perp = N\,dt$ and thus vanish when restricted to the spatial hypersurface. Reverting to tensor notation with a coordinate basis, the first term is

$$\Delta\omega^i{}_j \wedge \eta_i{}^j{}_\perp = \Delta\Gamma^i{}_{jk} g^{jm}\delta^{lk}_{im}\sqrt{g}dS_l = (g^{ij}g^{kl} - g^{il}g^{kj})\mathring{\nabla}_i\Delta g_{kj}\sqrt{g}dS_l\,, \tag{55}$$

where we have used

$$\Delta\Gamma^i{}_{kl} = \tfrac{1}{2}g^{jm}(\mathring{\nabla}_k\Delta g_{ml} + \mathring{\nabla}_l\Delta g_{mk} - \mathring{\nabla}_m\Delta g_{kl})\,. \tag{56}$$

Similarly, the second term can be rewritten in terms of the extrinsic curvature and then in terms of the ADM conjugate momentum:

$$\begin{aligned}
2\Delta\omega^{\perp j} \wedge N^k\eta_{\perp jk} &= -2\Delta(K^j{}_l\,dx^l) \wedge N^k\eta_{jk\perp} = -2N^k\Delta K^j{}_m\,\delta^{im}_{jk}\sqrt{g}dS_i \\
&= -2N^k\Delta(K^i{}_k - \delta^i_k K^j{}_j)\sqrt{g}dS_i = -2N^k\Delta(-g^{-1/2}\Pi^{ij}g_{jk})\sqrt{g}dS_i\,.
\end{aligned} \tag{57}$$

To appreciate the significance of the remaining term, we consider our boundary to be at spatial infinity. Asymptotically, with $\mathring{N}^j$ constant in time and the usual fall-offs, the final term takes the form

$$\mathring{D}^j \mathring{N}^\perp \,\Delta\vartheta^k \wedge \eta_{\perp jk} = g^{jl}N_{,l}\,\tfrac{1}{2}\,g^{kn}\Delta g_{ni}\,\delta^{mi}_{jk}\,\sqrt{g}\,dS_m\,; \tag{58}$$

here we have made the acceptable assumption that $(\Delta\vartheta^k)_m$ is symmetric and hence determined by $(1/2)\Delta g_{ij}$. Taking it all together, at spatial infinity our boundary term reduces to

$$\oint dS_l \left[\sqrt{g}(g^{ij}g^{kl} - g^{il}g^{kj})\,(N\mathring{\nabla}_i\Delta g_{kj} + N_{,i}\Delta g_{kj}) + 2N^i\Pi^{kl}g_{ik}\right], \tag{59}$$

where $\mathring{g}_{ij}$ and $\mathring{\nabla}_i$ are flat space quantities. This is just the Hamiltonian boundary term (generalized to a form suitable for computation in non-cartesian coordinates) found by

Beig and Ó Murchadha [38] to be necessary to allow for Poincaré displacements of the asymptotic Minkowski structure at spatial infinity for the asymptotically flat Einstein theory. The novel feature which they discovered is the $N_{,i}\Delta g_{jk}$ term, which is, according to their analysis, absolutely necessary to account for the boosts. This is where the Δ in our version of the "Møller-Komar" term affects the calculated values. Concerning the value of these boundary terms, the N and N^k terms give the ADM energy and momentum respectively, the N^k term also gives the angular momentum. Note that the $N_{,k}$ term makes no contribution unless the lapse N is asymptotically unbounded.

More recently Katz, working with Bičák, Lerer and Lynden-Bell, has generalized his previously mentioned work, using Noether conservation arguments applied to a Lagrangian quadratic in the first derivatives of the metric along with a global (possibly curved) background, to obtain a boundary expression for conserved quantities which can be put in the very succinct form

$$2\{\sqrt{-g}\,N^{[\rho}K^{\tau]} + \Delta(\sqrt{-g}\,D^{[\rho}N^{\tau]})\}. \tag{60}$$

Via manipulations such as those used in conjuction with eq (52) this is *exactly* the same as our expression $\mathcal{B}_g^{\tau\rho}$. We regard this is a significant support for our particular GR results and thus indirectly for our general ideas (conversely, our independent approach also supports the work of these other investigators). It seems that our nearly simultaneous efforts are rather complimentary. Consequently the successes achieved by these other workers in applications as diverse as Mach's principle [63], conservation laws at null infinity [64] and integral constraints for large cosmological perturbations [65] are further conformations for our work. This interesting body of work has only recently come to our attention; unfortunately, perusal of the connections between our work and that of Katz et al and the applications suggested thereby cannot be developed further at this time.

The work just mentioned, however, brings up an alternate approach to our whole analysis of quasilocal quantities. Instead of adjusting the Hamiltonian, we could have begun with an adjustment to the Hilbert Lagrangian by a boundary term. Our technique gives essentially only two different covariant versions for the GR Lagrangian with boundary term:

$$\mathcal{L}_\omega := \Omega^\alpha{}_\beta \wedge \eta_\alpha{}^\beta - d(\Delta\omega^\alpha{}_\beta \wedge \mathring{\eta}_\alpha{}^\beta)\,, \tag{61}$$

$$\mathcal{L}_\vartheta := \Omega^\alpha{}_\beta \wedge \eta_\alpha{}^\beta - d(\Delta\omega^\alpha{}_\beta \wedge \eta_\alpha{}^\beta)\,. \tag{62}$$

Let expressions with a tilde denote the quantity with a trivial reference configuration, then

$$\tilde{\mathcal{L}}_\vartheta := \Omega^\alpha{}_\beta \wedge \eta_\alpha{}^\beta - d(\omega^\alpha{}_\beta \wedge \eta_\alpha{}^\beta)\,. \tag{63}$$

The boundary term here is just the trace of the extrinsic curvature—this can be seen

by noting that the restriction of the boundary term to a surface with normal $e_\perp$ is

$$2\omega^\perp{}_a \wedge \eta_\perp{}^a = -2K_{ab}\vartheta^b \wedge \eta_\perp{}^a = -2K_{ab}g^{ab}\eta_\perp. \tag{64}$$

Hence we can identify $\tilde{\mathcal{L}}_\vartheta$ as the Lagrangian used by York and Brown [29, 14]; its space-time decomposition has the form

$$S^1 := \int_M \mathcal{L}_{\mathrm{BY}} = \frac{1}{2\chi}\int_M d^4x\sqrt{-g}R + \frac{1}{\chi}\int_{t'}^{t''} d^3x\sqrt{h}K - \frac{1}{\chi}\int_{^3B} d^3x\sqrt{-\gamma}\Theta\,, \tag{65}$$

where K and Θ are the traces of the extrinsic curvatures of the constant t and spatial boundary 3-surfaces which have metrics h_{ij} and γ_{ij}, respectively. Brown and York begin with this Lagrangian and, without discarding any total derivatives, construct the Hamiltonian along with its boundary term. Consequently, beginning with (63), a straightforward construction of the 'covariant Hamiltonian', without discarding any total derivatives, leads to their Hamiltonian boundary term expressed in terms of covariant quantities:

$$\tilde{\mathcal{B}}_{\mathrm{BY}}(N) = \omega^\alpha{}_\beta \wedge i_N\eta_\alpha{}^\beta. \tag{66}$$

By the same type of techniques as used above in connection with (54,57), it is easily verified that this 4-covariant expression (it's essentially just the Freud superpotential again) really does yield the densities for the quasilocal energy and momentum [14] which they derived from the physical Lagrangian S^1.

Brown and York took a Hamilton-Jacobi approach; they chose a reference action and considered $S = S^1 - S^0$, so their renormalized quasilocal quantities are of the form

$$\mathcal{B}_{\mathrm{BY}}(N) := \tilde{\mathcal{B}}_{\mathrm{BY}}(N) - \mathcal{B}^0_{\mathrm{BY}}(N)\,. \tag{67}$$

With a suitable choice for the reference action (it's necessary to use a reference 2-geometry which is isometric with the boundary 2-surface) we can obtain a covariant version of their expression:

$$\mathcal{B}_{\mathrm{BY}}(N) = \omega^\alpha{}_\beta \wedge i_N\eta_\alpha{}^\beta - \overset{\circ}{\omega}{}^\alpha{}_\beta \wedge i_N\eta_\alpha{}^\beta \equiv \Delta\omega^\alpha{}_\beta \wedge i_N\eta_\alpha{}^\beta. \tag{68}$$

Restricting to the 2-boundary, and using the previously referred to '3+1' decomposition techniques such as in (57) along with

$$\Delta\omega^i{}_j \wedge N^\perp\eta_i{}^j{}_\perp = 2N^\perp\Delta\omega^\vdash{}_A \wedge \eta_\vdash{}^A{}_\perp = 2N^\perp\Delta(-k_{AB}\vartheta^B) \wedge \eta_\vdash{}^A{}_\perp$$
$$= -2N^\perp\Delta(k_{AB}\sigma^{AB})\eta_{\vdash\perp}\,, \tag{69}$$

where $e_\vdash$, σ_{AB} are, respectively, the normal and metric (in an adapted frame) of the 2-boundary, plus a '2+1' decomposition of (57), leads to the quasilocal energy and momentum densities of Brown and York:

$$\epsilon = k|_0^{\mathrm{cl}}, \qquad j_i := -2(\sigma_{ik}(e_\vdash)_l\Pi^{kl}/\sqrt{h})|_0^{\mathrm{cl}}\,, \tag{70}$$

where k is the trace of the extrinsic curvature of the boundary 2-surface embedded in the constant t surface Σ. This covariant version of their expression is closely related to one of ours:

$$\mathcal{B}_{\mathrm{BY}}(N) \equiv \mathcal{B}_{\vartheta}(N) - i_N \mathring{\omega}^{\alpha}{}_{\beta} \Delta \eta_{\alpha}{}^{\beta} \,. \tag{71}$$

They exactly agree only when $\mathring{\omega}$ vanishes. Furthermore, the boundary term in the associated Brown-York Hamiltonian variation, $d(i_N(\Delta\omega^{\alpha}{}_{\beta} \wedge \delta\eta_{\alpha}{}^{\beta}) - i_N \mathring{\omega}^{\alpha}{}_{\beta} \delta\eta_{\alpha}{}^{\beta})$, does not have a covariant response unless $\mathring{\omega}$ vanishes. (In that case the reference configuration is flat Minkowski space.) From the Komar form of the difference term, we see that their quasilocal quantities will have the same values as ours as long as the shift is constant in time and the lapse is spatially constant.

Although our expressions are quite similar to these well known expressions of Brown and York, there are some important differences. In particular (i) our expressions are manifestly covariant, (ii) we do not require the boundary to be orthogonal to the spatial hypersurface (a simplifying restriction they had used which has recently been relaxed [66, 67, 68]), (iii) we consider more general reference configurations, (iv) our displacement vector field N is more general (as will be explained shortly), (v) our expressions have a "Komar like" DN term.

6.1. Static spherically symmetric solutions

As an example and in order to briefly address the issues of the selection of a reference configuration and displacement vector field, we consider certain simple solutions of general relativity: the static spherically symmetric metrics. We use the spherical coframe

$$\vartheta^t = \Phi dt, \quad \vartheta^r = \Phi^{-1} dr, \quad \vartheta^\theta = rd\theta, \quad \vartheta^\varphi = r \sin\theta \, d\varphi, \tag{72}$$

where $\Phi = \Phi(r)$. Specifically we have in mind the Schwarzschild (anti)-de Sitter and Reissner-Nordström type metrics:

$$\Phi^2 = 1 - \frac{2M}{r} + \lambda r^2 + \frac{Q^2}{r^2} \,. \tag{73}$$

In cases like this, where we have an exact analytic form for the metric which depends on a few parameters, there is an obvious simple choice for the reference configuration: just allow the parameters to have their trivial values. If $\lambda \neq 0$ we then have the choice of a Minkowski or anti-de Sitter reference space. However it is easy to see (with computations like those below) that the Minkowski choice will yield an energy which, for $\lambda \neq 0$, diverges as $r \to \infty$, so we report in detail only the anti-de Sitter choice:

$$\mathring{\vartheta}^t = \Phi_0 dt, \quad \mathring{\vartheta}^r = \Phi_0^{-1} dr, \quad \mathring{\vartheta}^\theta = rd\theta, \quad \mathring{\vartheta}^\varphi = r \sin\theta \, d\varphi, \tag{74}$$

where $\Phi_0^2 = 1 + \lambda r^2$. For the special case of $\lambda = 0$ the reference space reduces to Minkowski space.

Because of spherical symmetry, linear and angular momentum vanish, so we need only calculate the energy. By symmetry, the timelike displacement vector field should be orthogonal to the constant t spacelike hypersurfaces. Thus it should have the form $N = \alpha e_t$, which still allows for different "definitions" of energy. Our preferred choice is to define energy using the reference configuration timelike Killing vector $\partial_t = \Phi_0 \mathring{e}_t = \Phi e_t$, i.e., $\alpha = \Phi$. Other "obvious" candidates are $N = e_t$ and $N = \mathring{e}_t$ which correspond to the choices of lapse $\alpha = 1$ and $\alpha = \Phi_0^{-1}\Phi$, respectively. We will compare the results of these choices.

Under the conditions we are considering, $\Delta\eta_{tr}$ vanishes, $i_N\eta_\alpha{}^\beta = \alpha\eta_{t\alpha}{}^\beta$ and $i_N\mathring{\eta}_\alpha{}^\beta = \alpha\Phi^{-1}\Phi_0\mathring{\eta}_{t\alpha}{}^\beta$. Consequently

$$\mathcal{B}_\vartheta(N) = \frac{\alpha}{2\chi}\Delta\omega^\alpha{}_\beta \wedge \eta_{t\alpha}{}^\beta = \frac{\alpha}{\chi}(\Delta\omega^{r\theta} \wedge \eta_{tr\theta} + \Delta\omega^{r\varphi} \wedge \eta_{tr\varphi})$$
$$= \frac{2\alpha}{\chi}\left(\frac{\Phi_0 - \Phi}{r}\right)\vartheta^\theta \wedge \vartheta^\varphi = \frac{2\alpha}{\chi}(\Phi_0 - \Phi)\,r\sin\theta\,d\theta \wedge d\varphi\,, \tag{75}$$

which leads to the quasilocal energy

$$E_\vartheta(N) = \alpha\,r\,(\Phi_0 - \Phi)\,. \tag{76}$$

At the horizon ($\Phi = 0$) this reduces to $r\alpha\Phi_0$, giving

$$E_\vartheta(\mathring{e}_t) = E_\vartheta(\partial_t) = 0, \qquad E_\vartheta(e_t) = r\Phi_0. \tag{77}$$

At large distances E_ϑ approaches $(\alpha/\Phi_0)(M - Q^2/2r + \Phi_0^{-2}M^2/2r)$. Hence, for $\lambda \neq 0$, the asymptotic result is

$$E_\vartheta(\partial_t) = M - Q^2/2r, \qquad E_\vartheta(e_t) = E_\vartheta(\mathring{e}_t) = 0, \tag{78}$$

while for $\lambda = 0$

$$E_\vartheta(e_t) = M - \frac{Q^2 - M^2}{2r}, \qquad E_\vartheta(\partial_t) = E_\vartheta(\mathring{e}_t) = M - \frac{Q^2 + M^2}{2r}. \tag{79}$$

On the other hand, for the other control mode

$$\mathcal{B}_\omega(N) = \frac{\alpha}{2\chi}\Delta\omega^\alpha{}_\beta \wedge \Phi^{-1}\Phi_0\mathring{\eta}_{t\alpha}{}^\beta = \frac{\alpha\Phi_0}{\chi\Phi}(\Delta\omega^{r\theta} \wedge \mathring{\eta}_{tr\theta} + \Delta\omega^{r\varphi} \wedge \mathring{\eta}_{tr\varphi})$$
$$= \frac{2\alpha\Phi_0}{\chi\Phi}\left(\frac{\Phi_0 - \Phi}{r}\right)\mathring{\vartheta}^\theta \wedge \mathring{\vartheta}^\varphi = \frac{2\alpha\Phi_0}{\chi\Phi}(\Phi_0 - \Phi)\,r\sin\theta\,d\theta \wedge d\varphi\,, \tag{80}$$

leads to the quasilocal energy

$$E_\omega(N) = \alpha\,r\,\Phi^{-1}\Phi_0(\Phi_0 - \Phi)\,. \tag{81}$$

At the horizon this is $r\alpha(\Phi_0^2/\Phi)$, giving

$$E_\omega(e_t) = \infty, \quad E_\omega(\partial_t) = r\Phi_0^2 = 2(M - Q^2/2r), \quad E_\omega(\mathring{e}_t) = r\Phi_0, \tag{82}$$

while for very large r it approaches $(\alpha/\Phi)(M - Q^2/2r + \Phi_0^{-2}M^2/2r)$, giving, for $\lambda \neq 0$,

$$E_\omega(\partial_t) = M - Q^2/2r, \qquad E_\omega(\mathring{e}_t) = E_\omega(e_t) = 0, \tag{83}$$

and, for $\lambda = 0$,

$$E_\omega(\mathring{e}_t) = E_\omega(\partial_t) = M - \frac{Q^2 - M^2}{2r}, \qquad E_\omega(e_t) = M - \frac{Q^2 - 3M^2}{2r}. \tag{84}$$

Because of the vanishing values at large distances for $\lambda \neq 0$, e_t and $\mathring{e}_t$ seem to be unsuitable choices. Then the vanishing of $E_\vartheta(\partial_t)$ at the horizon suggests that analytic matching may not be so physical.

A reasonable alternate way to determine the values of the reference quantities is by embedding a neighborhood of the boundary into a space which has the desired reference geometry and then pulling back all the quantities to the dynamic spacetime. In the present case we can assume that the reference geometry coframe has the same spherical form:

$$\mathring{\vartheta}^t = \Phi_0 dt', \quad \mathring{\vartheta}^r = \Phi_0^{-1} dr', \quad \mathring{\vartheta}^\theta = r' d\theta', \quad \mathring{\vartheta}^\varphi = r' \sin\theta' \, d\varphi', \tag{85}$$

where, in particular, $\Phi_0^2 = 1 + \lambda r'^2$. We first identify corresponding foliations by spacelike hypersurfaces. A good criteria for this identification is to relate points that have the same trace for the extrinsic curvature. In this particular case this criteria is satisfied by the obvious choice: the one form dt' corresponds to βdt for some function β. Within each spacelike hypersurface a neighborhood of the (assumed to be spherical) boundary $S = \partial\Sigma$ is diffeomorphic to a suitable image in the reference geometry. A reasonable choice for the embedding is to require that the corresponding 2 spheres have the same *intrinsic* geometry. The uniqueness of such an identification has been discussed in [14]. (One alternative would be to try to match the *extrinsic* geometries.) In the present case it simply means that $r = r'$, $\theta = \theta'$, $\varphi = \varphi'$. To completely fix the embedding we need to specify the corresponding timelike unit. An obvious simple choice is to take $t = t'$, i.e., $\beta = 1$. That just gives the analytic matching case already considered. On the other hand, giving due consideration to the operational measurement procedure, it seems more physically reasonable as well as more geometric to match corresponding units of *proper* time rather than coordinate time. Thus we can choose to fix our relative time coordinates by identifying ϑ^t and $\mathring{\vartheta}^t$ on $\partial\Sigma$, consequently $\beta = \Phi\Phi_0^{-1}$. It should be noted, however, that this choice *is not* integrable. This type of geometric matching is *instantaneous*. At a later instant of time the procedure will determine a new reference configuration. It cannot be expected that these instantaneous reference configurations will mesh to form a single reference geometry.

Proceeding with our quasilocal calculations as before, again, because of spherical symmetry we need only calculate the energy and different choices for the displacement vector field give different "definitions" of the energy. Our preferred choice is to use the

reference configuration timelike Killing vector $\partial_{t'}$ which corresponds in this case to the choice of lapse $\alpha = \Phi_0$. Other obvious candidates are $N = \partial_t$ and $N = e_t = \mathring{e}_t$ which correspond respectively to the choices of lapse $\alpha = \Phi$ and $\alpha = 1$. We will now compare the results of these choices. Under the conditions we are considering, $\Delta\eta_{tr}$ vanishes and $i_N\eta_\alpha{}^\beta = \alpha\eta_{t\alpha}{}^\beta = \alpha\mathring{\eta}_{t\alpha}{}^\beta = i_N\mathring{\eta}_\alpha{}^\beta$ so

$$\mathcal{B}_\omega = \mathcal{B}_\vartheta = \frac{\alpha}{2\chi}\Delta\omega^\alpha{}_\beta \wedge \eta_{t\alpha}{}^\beta = \frac{\alpha}{\chi}(\Delta\omega^{r\theta} \wedge \eta_{tr\theta} + \Delta\omega^{r\varphi} \wedge \eta_{tr\varphi})$$
$$= \frac{2\alpha}{\chi}(\frac{\Phi_0 - \Phi}{r})\,\vartheta^\theta \wedge \vartheta^\varphi = \frac{2\alpha}{\chi}(\Phi_0 - \Phi)\,r\sin\theta\,d\theta \wedge d\varphi\,. \tag{86}$$

Consequently the quasilocal energy is

$$E(N) = E_\vartheta(N) = E_\omega(N) = \alpha\,r\,(\Phi_0 - \Phi)\,. \tag{87}$$

For very large r this approaches $\alpha(\Phi_0^{-1}(M - Q^2/2r) + \Phi_0^{-3}M^2/2r)$ while at the horizon ($\Phi = 0$) it reduces to $\alpha r\Phi_0$. Hence, for asymptotically Minkowski spaces ($\lambda = 0$, $\Phi_0 = 1$) our expressions yield

$$E(\partial_{t'}) = E(e_t) = M + \frac{M^2 - Q^2}{2r}\,, \qquad E(\partial_t) = M - \frac{M^2 + Q^2}{2r}\,, \tag{88}$$

for very large r and

$$E(\partial_{t'}) = E(e_t) = r, \qquad E(\partial_t) = 0, \tag{89}$$

at the horizon. For asymptotically anti-de Sitter spaces our expressions yield

$$E(\partial_{t'}) = E(\partial_t) = M - \frac{Q^2}{2r}\,, \qquad E(e_t) = M\Phi_0^{-1} \to 0, \tag{90}$$

for very large r and

$$E(\partial_{t'}) = r\Phi_0^2, \qquad E(\partial_t) = 0, \qquad E(e_t) = r\Phi_0, \tag{91}$$

at the horizon.

We have proposed a reference configuration Killing field as a good choice for the displacement vector field. The above calculations show that for geometric matching this choice gives reasonable values in practice. Note that the choice $N = \partial_t$ gives a vanishing value for the quasilocal energy within the horizon. For asymptotically flat space our spherically symmetric quasilocal energy values are the same as those found by Brown and York from their Hamilton-Jacobi approach. More generally, our proposal for the evolution vector field relaxes their choice of $\alpha = 1$. Consequently we get the total energy M for asymptotically anti de-Sitter solutions whereas their choice of $\alpha = 1$ leads to $E = M(1 + \lambda r^2)^{-1}$ which vanishes asymptotically. For the Hamilton-Jacobi approach to anti de-Sitter space see [69]

A direct consequence of our favored choice of the reference configuration timelike Killing vector is that, just as for Brown and York, the quasilocal energy for the

Schwarzschild solution is $2M$ at the horizon and M at infinity. Thus between the horizon and spatial infinity the quasilocal energy is *negative*, contrary to a property favored by several investigators [17, 9, 10]. However, as G. Hayward [70] has observed, the quasilocal energy *cannot* be positive in general—simply because closed spaces must have zero total energy so they must have negative regions to balance the positive regions.

The result of Landau-Lifshitz, Tolman [71] and S. Hayward [13] for the Reissner-Nordström metric is $E = M - Q^2/2r$. They find that the gravitational field has a remarkable difference from the electromagnetic field. The energy of the electromagnetic field is shared by the interior as well as the exterior of the horizon of the system, but the gravitational field energy is confined to its interior only. However, in our result the distribution of the gravitational and electromagnetic energy contributions is similar. Both are shared by the interior and exterior. One again this is a consequence of our choice for N. For the $Q = 0$ case, some, e.g. [17], have argued that the quasilocal energy for spherically symmetric solutions should simply be M independent of r. In our formulation that can be achieved simply by choosing the lapse to have a suitable value. (The special orthonormal frame Hamiltonian [72] also naturally suggests a choice which "localizes" all of the mass within the horizon.) More generally we can obtain the aforementioned value $E = M - Q^2/2r$ simply by choosing a (rather strange) value for the lapse such as $\alpha = (\Phi_0 + \Phi)/2$. By a judicious choice of the reference configuration as well as the lapse our expressions can yield results matching most of the values (e.g., those in [73]) found by other investigators for spherically symmetric systems.

7.   Discussion

The quest for gravitational energy-momentum has led to the quasilocal idea: *quasilocal quantities*, including energy-momentum and angular momentum, are associated with a closed 2-surface. In this work we have taken a Hamiltonian approach, defining the quasilocal quantities in terms of the value of the Hamiltonian for a finite region. This value is completely determined by the Hamiltonian boundary term.

We have considered a rather general class of geometric gravity theories, namely those having an independent metric and connection (Metric-Affine gravity). The field equations were presumed to follow from a Lagrangian which depends on the metric, torsion and curvature. (However, there is no real barrier to extending the procedure to include derivatives of these fields.) In our approach, special restrictions including metric compatibility, a symmetric connection or teleparallel geometry are easily introduced to lead to the familiar theories of interest to most investigators.

For this general class of geometric gravity theories, by using a covariant canonical Hamiltonian formalism, we presented both a general procedure, based on the boundary term in the variation of the Hamiltonian and its symplectic structure, and a specific

proposal: that this term should have a well defined *covariant* symplectic structure. We found that, for each dynamic field, there is a choice between two covariant boundary terms (essentially Dirichlet or Neumann boundary conditions). We then applied our formalism to black hole thermodynamics obtaining thereby an expression for the entropy in these general theories. Then we restricted our expressions to general relativity and applied them to well known spherically symmetric solutions, comparing the results with those found by some other authors. (Remarkably one of our expressions turns out to be equivalent to an expression recently developed and applied by Katz and his coworkers.) An important omission, in the context of Einstein's theory, is the relationship of our formalism to spinor formulations. This topic will be considered in a follow up work [74]. Here we wish to further discuss several issues that have arisen.

From a physical ("operational") point of view, suppose some observers attempt to measure the quasilocal quantities. What kind of procedure could they use? What will the result depend on? Ideally it should be purely geometric — depending only on the dynamical variables at the boundary, independent of any reference frame or choice of coordinates. Careful analysis, however, has led us to the recognition that specifying a quasilocal gravitational energy-momentum requires many choices, in particular

(i)   the theory: e.g., GR, Brans-Dicke, Einstein-Cartan, PGT, MA (Metric-Affine [42]);

(ii)   the representation or dynamic variables: e.g., the metric, orthonormal frame, connection, spinors;

(iii)   the control mode or boundary conditions: e.g., covariant, Dirichlet/Neumann for each dynamic variable;

(iv)   the reference configuration: e.g., Minkowski;

(v)   a displacement vector field: we propose using a Killing field of the reference geometry.

In this investigation we have determined how the quasilocal expressions should depend on these choices.

Given a specific theory and representation our formalism determines the covariant quasilocal expressions uniquely, however we do not yet get a unique value for the quasilocal quantities in a finite region. This is because there are two features still to be determined. One is the displacement vector N, and the other is the reference configuration.

The most ambiguous part of our program is the reference configuration. (We prefer the more general term "reference configuration" as opposed to "reference geometry" simply because the same type of analysis applies to other dynamic fields, e.g., electromagnetism.) It can not be avoided in our expressions. Operationally it can be related to the calibration or "renormalization" of our measurement. If the dynamical variables take on the reference values our expressions will give zero values for all quasilocal quantities.

How to choose an appropriate reference configuration? We are not really satisfied with our understanding of this topic but will share our current thoughts here. Since our boundary could be anywhere we could simple use a globally fixed background geometry. For an asymptotically flat or constant curvature spacetime, we can choose the simplest global spacetime (let the source vanish in the original spacetime) which has the same asymptotic behavior as the physical spacetime. This is not so ambiguous asymptotically; it allows one to determine the total conserved quantities. But a global background is hardly necessary, moreover it is operationally impractical. How then should one determine the reference configuration for a finite region?

We can imagine the dynamic fields in the neighborhood of the boundary as deviating from some "ground state" configuration. These "reference values" can be viewed as defining (at least a portion of) a conceptually distinct geometric reference space. A slight shift of view then leads to a simple method (which we have elaborated upon) for constructing the reference configuration for the geometric quantities of interest here: use an embedding (this depends on some matching criteria such as proper time and intrinsic 2-surface isometry) of a certain neighborhood of the topologically compact orientable boundary spatial 2-surface (usually with spherical topology) into a space having the desired reference geometry (ordinarily a Minkowski geometry, but alternatives include Schwarzschild, (anti) de Sitter, or a homogeneous cosmology). Pull back the geometry to determine the reference configuration. (At the same time we can also pull back a displacement vector.) The important question is whether we can find a unique embedding? (And, of course, what does it mean physically?) Brown and York have proposed embedding the 2-surface isometrically [14]. They have referred to some uniqueness theorems for the embedding of topologically spherical positive curvature 2-metrics. Such an embedding seems like a good idea. However, we do not yet adequately understand what this means physically or whether some other idea would be equally reasonable.

Another way to deal with the reference configuration is to replace it with some additional covariant field satisfying some propagation equation. To our knowledge this approach has so far only been used in the context of spinor expressions [75, 76]. In this case the propagation equation, in effect, implicitly determines the reference configuration. For example, the Dougan and Mason spinor propagation equation [9] determines a spinor field on a closed 2-surface. There is a good prospect for relating this approach and the geometric embedding approach. For example a spinor field satisfying a suitable propagation equation could be used to determine an orthonormal frame which, in turn, could be used to determine a reference configuration for the geometrical quantities.

Turning to the displacement vector N (which corresponds to the *lapse function* and the *shift vector* in the ADM analysis [77]). How should we choose the appropriate

displacement vector in order to get the "proper" quasilocal quantities? For the total conserved quantities, the choice is unique, namely an asymptotic Killing vector. But for the quasilocal quantities in a finite region, the corresponding Killing vector may not exist. Obviously energy is related to time translation, momentum to space translation and angular momentum to rotation but which precise choice of timelike vector field gives the energy? For example the Brown and York [14] choice for energy, lapse = 1 and shift = 0, seems natural. Yet with it, for asymptotically anti-de Sitter solutions, the quasilocal energy will not converge to the total energy. The choice of N is probably best tied to the choice of reference configuration. We have proposed using a Killing field of the reference configuration to define quasilocal energy, momentum and angular momentum. As in [14] alternate choices for the displacement vector field could be used to distinguish a quasilocal quantity from a conserved charge. For example, each Killing field of the dynamic spacetime would give a conserved charge.

The quasilocal expressions that we found are well behaved and are related to expressions found by other investigators. They satisfy the usual criteria: their values have good correspondence limits to flat space and spherically symmetric values, with good weak field and asymptotic limits for both asymptotically flat and constant curvature spaces, and in both the Bondi and ADM limit. (However a complete analysis of the asymptotics which will give well defined quasilocal quantities and the necessary conditions for the Hamiltonian to be a differential generator in the sense of [78] remains to be completed.) Although our expressions mesh with a positive total energy proof for asymptotically flat solutions to Einstein's theory, our expressions are not locally positive in general. However, we regard this as a virtue rather than a drawback for they thus allow for the correct (vanishing) total for closed spaces. In any case, such criteria are known to be insufficient.

We have emphasized an additional principle of the Hamiltonian formalism concerning the vanishing of the boundary term in the variation of the Hamiltonian for suitable boundary conditions. Specifically we have required that the boundary term in the Hamiltonian variation have a well defined *covariant symplectic* structure.

We stated that we have determined the expressions for quasilocal quantities uniquely. Here the meaning of the word *uniquely* is based on the requirement of the covariant symplectic structure. We could release this rather strong requirement to a weaker version which only requires a well defined symplectic structure. For example, for a vector field W^μ, we could use Dirichlet conditions on part of the spacetime components say W^0, W^1 and Neumann conditions on the remaining components. Then many more quasilocal expressions are possible. This is just the sort of thing that happens with the i_N factor in the boundary term in the Hamiltonian variations (17,19).

From the physical point of view, certain noncovariant choices might be distinguished and may sometimes be more practical. One approach is to consider the symplectic

structure, taking into account the initial value constraints or the boundary conditions needed for a good initial value problem. (Boundary control or fixed on the boundary, by the way, does not mean having a constant value, but rather means that the value of the variable is a preprogrammed function of time.) In particular, for the Einstein theory, Jezierski and Kijowski [8] have investigated decomposing the variables into the true unconstrained degrees of freedom. They considered the form of the initial value constraints and found that for certain control modes the boundary value problem was not well posed. From a naive view point, the symplectic structure reflects the control-response relation of the physical system. So for some specific theory, the symplectic structure may need some modifications, because the constraints could forbid some control mode at the boundary. In principle, we could handle such constraints by imposing them with Lagrange multiplier terms in the variational principle. This may modify the symplectic structure and the expressions for quasilocal quantities, effectively automatically including the constraints. Further investigation is certainly needed on this topic.

Others boundary control choices might be more closely related to what an observer would directly measure. For example, in an electromagnetic system, we can naturally fix the electric potential and the tangential components of the magnetic field instead of the whole 4-vector potential at the boundary, or we can fix the normal component of the electric field and the tangential components of the vector potential (essentially the normal component of the magnetic field). The physical meaning as well as the procedure for implementing these choices is well understood. But the boundary term in the corresponding Hamiltonian variation for such control modes depends on the fields in a noncovariant way. However, these non-covariant expressions are related to a covariant expression of our type by a simple Legendre transformation on the boundary. Having measured the values for one of these situations, we could calculate the value of our covariant quasilocal expressions even if they could not be conveniently measured directly.

We have emphasized the importance of covariance for good physical quasilocal quantities. Yet here exists a noncovariant term, $i_N \omega^\alpha{}_\beta \Delta \rho_\alpha{}^\beta$, in our Hamiltonian boundary expressions. This term merits deeper consideration. We noted that it was through this term that our expressions mainly differed from those found by others. Consequently, further investigation is desirable to ascertain its importance and how essential it is that it have exactly the form that we found. We have decomposed this term by using the identity (24). The first two terms on the rhs of this identity are covariant and represent a real physical effect. The last term, which reflects a noncovariant property of our expressions, is an unphysical term presenting a dynamical reference frame effect. This situation is like the centrifugal force in mechanics. The complete Hamiltonian generates the (arbitrary) dynamical evolution of the frame (effectively, the observer) as well as the physical variables. Through the relation (24), the complete Hamiltonian

implicitly includes a time derivative term on the boundary. Within the context of Einstein's theory, boundary time derivative terms have been noted some time ago by Kijowski [27] and more recently by G. Hayward [70].

A major virtue of our formalism is that it provides for system and order. Although it allows for many "energies", yet each has (in principle) a clear physical meaning. The formalism systematically associates a quasilocal energy-momentum with a specific Hamiltonian boundary term (determined by the choice of theory, representation— dynamic variables, control mode, reference configuration, matching criteria, and displacement vector field). The gravitational quasilocal energy-momentum is thus, like the energies in thermodynamics, connected with a definite physical situation with definite conditions on the boundary. But much more investigation is needed to understand the significance of various boundary conditions. Hence our formalism shifts the attention to questions like "What is the real physical significance of controlling the metric (or the connection) on the boundary of a finite region? Which boundary control (if any) is the analog of thermally insulating a system? Which corresponds to a thermal bath?" Some insight is afforded by classical electrodynamics where the answers to such questions are understood [27, 8].

Of course one way to further investigate the various quasilocal expressions is to do more direct calculations for exact solutions, e.g., [79]. However, a deeper theoretical investigation should be more revealing. Our formulation provides a good starting point for such an investigation. Note that *all* of the expressions presented here correspond to the work done in some (ideal) physical process. The situation is similar to thermodynamics with its different energies (enthalpy, Gibbs, Helmholtz, etc.) An even better analogy is the electrostatic work required while controlling the potential on the boundary of a region vs. that required while controlling the charge density [35].

However, we have gone far enough for the present. Our intention here was primarily to set up a framework, to indicate something of its scope and the efficacy of the principles. In follow up works we plan to extend these investigations in several directions including the application to and relationship with spinor expressions for quasilocal quantities [74].

Acknowledgments

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